Representation Theory of Finite Groups
Mid-Semestral Exam
Feb. 2024

Time: 2 hours

Max score: 30

Notations: G denotes a finite group throughout, and all representations are over the field of complex numbers.

Answer all questions. Give complete justifications to your answers.

(1) (a) Let $\phi : G \to GL_3(\mathbb{C})$ be a representation of G. Show that ϕ is irreducible if and only if there is no common eigenvector for the matrices ϕ_g with $g \in G$. (4)

(2) (a) State and prove Schur's lemma.
(b) Use (a) to show that if ϕ is an irreducible representation of G, and ρ is any representation of G, then the multiplicity of ϕ in ρ is dimension of the space $\mathrm{Hom}_G(\phi, \rho)$. (4+4)

(3) (a) Define regular representation of a group G.
(b) Find the character of the regular representation.
(c) Show that if $\phi^1, \dots, \phi^s$ denote a complete set of representatives of the equivalence classes of irreducible representations of G, then the multiplicity of ϕ^i in the decomposition of the regular representation is $\deg(\phi^i)$, for all $1 \leq i \leq s$. (1+2+2)

(4) (a) Let ϕ and ψ be two representations of G. Show that ϕ is equivalent to ψ if and only if their characters are equal.
(b) Show that if L is the regular representation of G, and if $\rho : G \to GL(V)$ is any representation, then $\rho \otimes L$ is equivalent to $L^{\oplus \dim V}$, where $L^{\oplus \dim V}$ denotes the direct sum representation of $\dim V$ copies of L. (4+2)

(5) Write down the character table for S_4, the symmetric group on 4 elements. (7)